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Reversal of secondary flow in non-Newtonian fluids

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A study is made of the steady radial–axial flow generated in a non-Newtonian fluid confined between two infinite parallel planes by torsional oscillations of one of the planes. Canonical equations of motion appropriate for the small amplitude motion of a simple fluid are used. It is shown that under certain circumstances the direction of the radial–axial flow is opposite to that in a Newtonian fluid.

1. Introduction

Consider a fluid occupying the region between two infinite, parallel, rigid plane boundaries. Suppose that one of the planes performs small amplitude torsional oscillations about an axis normal to itself, while the other plane remains at rest. Clearly the motion of the boundary will induce flow of the fluid; the aim of this paper is to examine the nature of this flow.

In the case of a Newtonian fluid the problem was studied by Rosenblat (1959). It was shown that to first order in the amplitude the flow is purely azimuthal and has essentially the structure of a Stokes layer in which the velocity decays exponentially with distance from the boundary. At second order there is a radial–axial motion of the fluid, consisting of a steady component and a periodic component of twice the frequency of the forced oscillations. This secondary flow is generated primarily by centrifugal forces; the steady radial–axial movement is radially outward near the oscillating plane and radially inward near the stationary plane.

A number of authors subsequently considered generalizations of the problem to the case where the fluid was non-Newtonian. Thus Bhatnagar & Rajeswari (1962) and Srivastava (1963) obtained solutions for fluids which obey a second-order Rivlin–Ericksen constitutive equation, while Frater (1964) studied an elastico-viscous fluid satisfying an Oldroyd-type constitutive equation. These authors showed that the general flow structure described above (first-order azimuthal flow, second-order radial–axial flow) was present in each of their non-Newtonian fluid models. An interesting effect noticed by Bhatnagar & Rajeswari (1962) and by Frater (1964) was that in certain circumstances the direction of the steady component of the radial–axial flow was reversed by comparison with that in a Newtonian fluid.

Reversal of the direction of secondary flow is regarded as being one of the phenomena which characterize non-Newtonian behaviour in fluids (cf. Bird 1976). There are several reports in the literature of observations of secondary-flow reversal when the primary (azimuthal) flow is time independent. A particularly interesting account of recent experiments is given by Hill (1972), and a theoretical analysis of these experiments has been provided by Kramer & Johnson (1972). Both the theory and the experiments reported in these papers relate to a fluid contained in a circular cylinder, with its upper surface in contact with a steadily rotating disk. Thus the work of these authors concerns a geometrical configuration which is similar to ours except that the fluid in their case is bounded laterally rather than being of infinite extent. We shall see below that the lateral boundary makes an important difference to the behaviour of the flow.

Recently Chang & Schowalter (1974) have described experiments on acoustic streaming induced by the small oscillations of a cylinder in a viscoelastic fluid. They have observed that the direction of the steady streaming is opposite to that in a Newtonian fluid. These experiments are particularly interesting from our present point of view, since there is a well-known analogy between acoustic streaming and steady radial–axial motion generated by torsional oscillations.

In this paper we shall consider the problem described at the beginning of this section for a general simple fluid undergoing motions of small amplitude. The constitutive equations, derived originally by Coleman & Noll (1961), are essentially the equations of second-order viscoelasticity. They can be shown to be valid for every simple fluid with fading memory whose defining functional is sufficiently smooth and whose motion is such that the history tensor is small in norm. The explicit equations of motion are presented in § 2, and the solution is obtained and discussed in §§ 3–5.

Our work differs from that of Bhatnagar & Rajeswari (1962) and Frater (1964) in several respects. First, we use different constitutive equations, which we believe to be of greater generality. In particular, the use of the second-order Rivlin–Erickson model to describe time-dependent flows is now considered to be of doubtful validity in view of the fact that the rest state of such a fluid can be unstable (Craik 1968). Second, there is a discrepancy between the conclusions of Bhatnagar & Rajeswari (1962) and Frater (1964). The former authors find that flow reversal will always occur at sufficiently high frequency, while Frater (1964) obtains flow reversal only at intermediate frequencies. Our own conclusions are somewhat at variance with both of these; we feel, however, that our results are amenable to simple experimental verification. We give, moreover, a more detailed description of how flow reversal occurs than has been given in either of the papers cited.

2. Equations of motion

In this section we shall set out the equations governing the small amplitude time-dependent motions of a class of incompressible simple fluids. Our constitutive equations will apply to simple fluids whose defining functional possesses a certain smoothness property; in particular they will be appropriate for so-called simple fluids of integral type (Truesdell & Noll 1965, p. 98).

Let $\mathbf{x}$ be the position vector of a material point X at time t (which is regarded as being the present time). Let $\boldsymbol{\xi}^*$ be the position vector of X at time $\tau \leqslant t$. We introduce the variable s defined by $\tau = t-s$ and express the relationship between $\boldsymbol{\xi}^*$ and $\mathbf{x}$ through the formula

$$\boldsymbol{\xi}^* = \boldsymbol{\xi}^*(s) = \boldsymbol{\xi}^*(\mathbf{x}, t; t-s), \tag{2.1}$$

so that

$$\mathbf{x} = \boldsymbol{\xi}^*(t) = \boldsymbol{\xi}^*(\mathbf{x}, t; t). \tag{2.2}$$

Let $\mathbf{v}^*$ denote the velocity of the point X. Then we have that

$$d\boldsymbol{\xi}^*/ds = -\mathbf{v}^*(\boldsymbol{\xi}^*, t-s) \tag{2.3}$$

is the differential equation of the pathlines, and is subject to the initial condition

$$\boldsymbol{\xi}^* = \mathbf{x} \quad \text{when} \quad s = 0. \tag{2.4}$$

We define the tensor $\mathbf{F}(s)$ by the relation

$$\mathbf{F}(s) = \mathbf{F}(\mathbf{x}, t; t-s) = \nabla\boldsymbol{\xi}^*(s) = \nabla\boldsymbol{\xi}^*(\mathbf{x}, t; t-s). \tag{2.5}$$

Evidently this has the property that $\mathbf{F}(0) = \mathbf{I}$, where $\mathbf{I}$ is the unit tensor. From (2.5) we are able to define the history tensor $\mathbf{G}^*(s)$ by means of the formula

$$\mathbf{G}^*(s) = \mathbf{G}^*(\mathbf{x}, t; t-s) = \mathbf{F}^{\mathrm{T}}(s) . \mathbf{F}(s) - \mathbf{I}, \quad \mathbf{G}^*(0) = 0. \tag{2.6}$$

The constitutive equation for an incompressible simple fluid is taken in the form

$$\mathbf{T} + p\mathbf{I} = \mathbf{S}^* = \mathop{\mathscr{F}}_{s=0}^{\infty} (\mathbf{G}^*(s)), \tag{2.7}$$

where $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ is the stress, $p = p(\mathbf{x}, t)$ is the pressure, $\mathbf{S}^* = \mathbf{S}^*(\mathbf{x}, t)$ is the extra stress and $\mathscr{F}$ is a tensor-valued functional defined on a prescribed set of histories $\mathbf{G}^*$. This functional must satisfy an isotropy relation (Truesdell & Noll 1965, p. 78), and has the property that $\mathscr{F}(0) = 0$.

Now let ϵ be a small parameter, real and positive. In the general theory ϵ is a measure of the departure from the rest state of the fluid, uniformly in space and time; in the present situation it is understood to be a dimensionless amplitude of oscillation of the boundary. We expand the velocity field in the form

$$\mathbf{v}^*(\mathbf{x}, t) = \epsilon\mathbf{v}(\mathbf{x}, t, \epsilon) = \epsilon\mathbf{v}_1(\mathbf{x}, t) + \epsilon^2\mathbf{v}_2(\mathbf{x}, t) + O(\epsilon^3) \tag{2.8}$$

and introduce the vector $\boldsymbol{\xi} = \boldsymbol{\xi}(s, \epsilon) = \boldsymbol{\xi}(\mathbf{x}, t; t-s; \epsilon)$ defined by

$$\boldsymbol{\xi}^*(s) = \mathbf{x} + \epsilon\boldsymbol{\xi}(s, \epsilon) = \mathbf{x} + \epsilon\boldsymbol{\xi}_1(s) + \epsilon^2\boldsymbol{\xi}_2(s) + O(\epsilon^3). \tag{2.9}$$

Similarly we write

$$\mathbf{G}^*(s) = \epsilon\mathbf{G}(s, \epsilon) = \epsilon\mathbf{G}_1(s) + \epsilon^2\mathbf{G}_2(s) + O(\epsilon^3), \tag{2.10}$$

and it can be shown by reasonably direct computation that

$$\mathbf{G}_1(s) = \mathbf{G}_1(\mathbf{x}, t; t-s) = \nabla\boldsymbol{\xi}_1(s) + (\nabla\boldsymbol{\xi}_1(s))^{\mathrm{T}} \tag{2.11}$$

and

$$\mathbf{G}_2(s) = \mathbf{G}_2(\mathbf{x}, t; t-s) = \nabla\boldsymbol{\xi}_2(s) + (\nabla\boldsymbol{\xi}_2(s))^{\mathrm{T}} + (\nabla\boldsymbol{\xi}_1(s))^{\mathrm{T}} . \nabla\boldsymbol{\xi}_1(s). \tag{2.12}$$

The corresponding expansion for the constitutive relation (2.7) can be obtained for a general functional $\mathscr{F}$ which satisfies certain smoothness conditions. The precise requirements on $\mathscr{F}$ are to be found in Coleman & Noll (1961) and the detailed steps of the expansion procedure are set out in Joseph (1976, chap. 13). We confine ourselves therefore to a statement of the results, which are as follows. If the extra-stress tensor $\mathbf{S}^*$ is expanded in the form

$$\mathbf{S}^* = \epsilon\mathbf{S} = \epsilon\mathbf{S}_1 + \epsilon^2\mathbf{S}_2 + O(\epsilon^3), \tag{2.13}$$

then it can be shown that

$$\mathbf{S}_1 = \int_0^\infty \zeta(s)\, \mathbf{C}_1(s)\, ds \tag{2.14}$$

and

$$\mathbf{S}_2 = \int_0^\infty \zeta(s)\, \mathbf{C}_2(s)\, ds + \int_0^\infty \zeta(s) \{(\boldsymbol{\xi}_1 . \nabla)\, \mathbf{C}_1(s) + \mathbf{C}_1(s) . \nabla\boldsymbol{\xi}_1 + (\nabla\boldsymbol{\xi}_1)^{\mathrm{T}} . \mathbf{C}_1(s)\}\, ds + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, \mathbf{C}_1(s_1) . \mathbf{C}_1(s_2)\, ds_1\, ds_2, \tag{2.15}$$

where we have introduced the notation

$$\mathbf{C}_i(s) = \mathbf{C}_i(\mathbf{x}, t-s) = \nabla \mathbf{v}_i(\mathbf{x}, t-s) + (\nabla \mathbf{v}_i(\mathbf{x}, t-s))^{\mathrm{T}}, \quad i = 1, 2. \tag{2.16}$$

The functions $\zeta(s)$ and $\gamma(s_1, s_2)$ are material functions; the former is the stress relaxation modulus and the latter has the property that $\gamma(s_1, s_2) = \gamma(s_2, s_1)$.

Equation (2.14) is recognizable as the constitutive equation of linear viscoelasticity. Equation (2.15) represents a second-order viscoelastic effect, and was apparently first obtained in this form by Joseph (1976). It is fairly easy to show that for *steady* motions (2.13)–(2.16) reduce to the constitutive equations for a second-order Rivlin–Ericksen fluid. The Rivlin–Eriksen material constants, denoted by μ_0, α_1 and α_2, can be shown to be related to the material functions ζ and γ by the formulae

$$\left.\begin{aligned} \mu_0 = \int_0^\infty \zeta(s)\,ds, \quad \alpha_1 - = \int_0^\infty s\zeta(s)\,ds, \\ \alpha_2 = \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\,ds_1\,ds_2. \end{aligned}\right\} \tag{2.17}$$

In the absence of body forces the motion of the fluid is governed by the dynamical equations

$$\rho(\partial \mathbf{v}^*/\partial t + \mathbf{v}^*.\nabla \mathbf{v}^*) = -\nabla p^* + \nabla.\mathbf{S}^*, \tag{2.18}$$

where ρ is the density, together with the continuity equation

$$\nabla.\mathbf{v}^* = 0. \tag{2.19}$$

We substitute into these equations the expansions (2.8) and (2.13) for $\mathbf{v}^*$ and $\mathbf{S}^*$ respectively, and a similar expansion for p^*. Equating coefficients of like powers of ϵ we obtain the following system:

$$\rho\,\partial \mathbf{v}_1/\partial t = -\nabla p_1 + \nabla.\mathbf{S}_1, \quad \nabla.\mathbf{v}_1 = 0, \tag{2.20}$$

$$\rho(\partial \mathbf{v}_1/\partial t + \mathbf{v}_1.\nabla \mathbf{v}_1) = -\nabla p_2 + \nabla.\mathbf{S}_2, \quad \nabla.\mathbf{v}_2 = 0. \tag{2.21}$$

These are the governing equations for the problem considered in this paper, with $\mathbf{S}_1$ and $\mathbf{S}_2$ given by (2.14) and (2.15) respectively.

3. The first-order flow

Take cylindrical polar co-ordinates (r, θ, z) and let the corresponding unit vectors be $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{z}}$. Let the fluid occupy the region between rigid planes located at $z = 0$ and $z = d$. We consider the flow generated by torsional oscillations of the plane $z = 0$ about the axis $r = 0$, while the plane $z = d$ remains at rest. In the notation of § 2 we wish to find a solution to (2.18) and (2.19) subject to the boundary conditions

$$\mathbf{v}^* = \epsilon r \cos \omega t\, \hat{\boldsymbol{\theta}} \quad \text{on} \quad z = 0, \quad \mathbf{v}^* = 0 \quad \text{on} \quad z = d. \tag{3.1}$$

Here the parameter ϵ has the dimensions of angular velocity.

It is easy to see that the problem reduces to the solution in sequence of the following problems:

(i) the system (2.20), with $\mathbf{S}_1$ given by (2.14), subject to the boundary conditions

$$\mathbf{v}_1 = r \cos \omega t\, \hat{\boldsymbol{\theta}} \quad \text{on} \quad z = 0, \quad \mathbf{v}_1 = 0 \quad \text{on} \quad z = d; \tag{3.2}$$

(ii) the system (2.21), with $\mathbf{S}_2$ given by (2.15), subject to the boundary conditions

$$\mathbf{v}_2 = 0 \quad \text{on} \quad z = 0, \quad \mathbf{v}_2 = 0 \quad \text{on} \quad z = d. \tag{3.3}$$

In this section we consider the first-order problem. We note from (2.16) that

$$\nabla . \mathbf{C}_1 = \nabla^2 \mathbf{v}_i, \quad i = 1, 2; \tag{3.4}$$

hence (2.20) and (2.14) reduce to the system

$$\rho \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \int_0^\infty \zeta(s) \nabla^2 \mathbf{v}_i \, ds, \quad \nabla . \mathbf{v}_1 = 0. \tag{3.5}$$

A solution compatible with the boundary conditions (3.2) is

$$\mathbf{v}_1 = r \operatorname{Re} \{\phi(z) e^{i\omega t}\} \hat{\boldsymbol{\theta}}, \quad p_1 = 0, \tag{3.6}$$

provided that $\phi(z)$ satisfies the boundary-value problem

$$d^2\phi/dz^2 - (i\omega\rho/\mu)\phi = 0, \quad \phi(0) = 1, \quad \phi(d) = 0, \tag{3.7}$$

where μ (sometimes called the complex viscosity) is defined by the formula

$$\mu = \mu(\omega) = \int_0^\infty \zeta(s) e^{-i\omega s} ds. \tag{3.8}$$

The solution of (3.7) is

$$\phi(z) = \frac{\sinh \frac{1}{2}\Omega(1 - z/d)}{\sinh \frac{1}{2}\Omega} \tag{3.9}$$

where

$$\Omega \equiv (4 i \rho \omega d^2/\mu)^{\frac{1}{2}} \tag{3.10}$$

may be regarded as a dimensionless (complex) frequency parameter. Hence we have the first-order flow field

$$\mathbf{v}_1 = r \operatorname{Re}\left\{\frac{\sinh \frac{1}{2}\Omega(1 - z/d)}{\sinh \frac{1}{2}\Omega} e^{i\omega t}\right\} \hat{\boldsymbol{\theta}}, \tag{3.11}$$

which is a well-known result.

We calculate the components of the stress tensor $\mathbf{S}_1$. From (3.6) we have that

$$\mathbf{v}_1 = \operatorname{Re}(\phi e^{i\omega t})(\hat{\boldsymbol{\theta}}\hat{\mathbf{r}} - \hat{\mathbf{r}}\hat{\boldsymbol{\theta}}) + r \operatorname{Re}(\phi' e^{i\omega t}) \hat{\boldsymbol{\theta}}\hat{\mathbf{z}}, \tag{3.12}$$

where $\phi' \equiv d\phi/dz$; on substituting this expression into (2.16) we obtain

$$\mathbf{C}_1(s) = r \operatorname{Re}\{\phi' e^{i\omega(t-s)}\}(\hat{\boldsymbol{\theta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\boldsymbol{\theta}}). \tag{3.13}$$

Equations (2.14) and (3.13) now give

$$\mathbf{S}_1 = r \operatorname{Re}\{\mu\phi' e^{i\omega t}\}(\hat{\boldsymbol{\theta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\boldsymbol{\theta}}). \tag{3.14}$$

The values of the shearing stresses at the bounding planes can be found by setting $z = 0$ and $z = d$ respectively in (3.14).

4. The second-order flow

We turn now to the calculation of the second-order flow. It is convenient to introduce complex-conjugate notation, and to write (3.13) in the form

$$\mathbf{C}_1(s) = \tfrac{1}{2} r \{\phi' \exp[i\omega(t-s)] + \bar{\phi}' \exp[-i\omega(t-s)]\}(\hat{\boldsymbol{\theta}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\boldsymbol{\theta}}). \tag{4.1}$$

From this we obtain

$$\mathbf{C}_1(s_1) \,.\, \mathbf{C}_1(s_2) = \tfrac{1}{4}r^2\{\phi'^2 \exp[i\omega(2t-s_1-s_2)] + \bar{\phi}'^2 \exp[-i\omega(2t-s_1-s_2)] + 2\phi'\bar{\phi}' \cos\omega(s_1-s_2)\}\,(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\mathbf{z}}\hat{\mathbf{z}}). \quad (4.2)$$

Next we have from (3.6) and integration of (2.3)

$$\boldsymbol{\xi}_1(\mathbf{x}, t;\, t-s) = -r\,\mathrm{Re}\int_0^s \phi(z)\exp[i\omega(t-s')]\,ds'\,\hat{\boldsymbol{\theta}} = r\,\mathrm{Re}\left\{\frac{i\phi(z)}{\omega}e^{i\omega t}(1-e^{-i\omega s})\right\}\hat{\boldsymbol{\theta}}. \quad (4.3)$$

From this formula and (4.1) we can compute the following result:

$$(\boldsymbol{\xi}_1 \,.\, \nabla)\,\mathbf{C}_1 + \mathbf{C}_1 \,.\, \nabla\boldsymbol{\xi}_1 + (\nabla\boldsymbol{\xi}_1)^{\mathrm{T}} \,.\, \mathbf{C}_1 = (ir^2/2\omega)\,\{\phi'^2 \exp[i\omega(2t-s)]\,(1-e^{-i\omega s}) - \bar{\phi}'^2 \exp[-i\omega(2t-s)]\,(1-e^{i\omega s}) + 2i\phi'\bar{\phi}'\sin\omega s\}\,\hat{\mathbf{z}}\hat{\mathbf{z}}. \quad (4.4)$$

It is evident from the forms of (4.2) and (4.4) that the second-order flow will comprise a steady component and a second-harmonic component. Thus we set

$$\mathbf{v}_2(\mathbf{x}, t) = \mathbf{v}_2^{(0)}(\mathbf{x}) + \tfrac{1}{2}[\mathbf{v}_2^{(2)}(\mathbf{x})\,e^{2i\omega t} + \bar{\mathbf{v}}_2^{(2)}(\mathbf{x})\,e^{-2i\omega t}], \quad (4.5)$$

$$p_2(\mathbf{x}, t) = p_2^{(0)}(\mathbf{x}) + \tfrac{1}{2}[p_2^{(2)}(\mathbf{x})\,e^{2i\omega t} + \bar{p}_2^{(2)}(\mathbf{x})\,e^{-2i\omega t}] \quad (4.6)$$

and

$$\mathbf{S}_2(\mathbf{x}, t) = \mathbf{S}_2^{(0)}(\mathbf{x}) + \tfrac{1}{2}[\mathbf{S}_2^{(2)}(\mathbf{x})\,e^{2i\omega t} + \bar{\mathbf{S}}_2^{(2)}(\mathbf{x})\,e^{-2i\omega t}]. \quad (4.7)$$

From (2.15) ,(2.16), (4.2) and (4.4) we can write down the explicit form of the steady part of $\mathbf{S}_2$. We find that

$$\mathbf{S}_2^{(0)}(\mathbf{x}) = \mu_0[\nabla\mathbf{v}_2^{(0)} + (\nabla\mathbf{v}_2^{(0)})^{\mathrm{T}}] + \beta_1(r^2\phi'\bar{\phi}')\,\hat{\mathbf{z}}\hat{\mathbf{z}} + \tfrac{1}{2}\beta_2(r^2\phi'\bar{\phi}')\,(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\mathbf{z}}\hat{\mathbf{z}}), \quad (4.8)$$

where μ_0 (the Newtonian viscosity) is given by (2.17), and where we have defined

$$\beta_1 = -\int_0^\infty \zeta(s)\frac{\sin\omega s}{\omega}\,ds \quad (4.9)$$

and

$$\beta_2 = \int_0^\infty\int_0^\infty \gamma(s_1, s_2)\cos\omega(s_1-s_2)\,ds_1\,ds_2. \quad (4.10)$$

Clearly as $\omega \to 0$ the parameters β_1 and β_2 reduce to the Rivlin–Eriksen constants α_1 and α_2 respectively.

In a similar way it is possible to write down the second-harmonic contribution to $\mathbf{S}_2$, but we shall not pursue the details here.

We can now state explicitly the equations and boundary conditions for the second-order fields. From (3.6) we have that

$$\mathbf{v}_1 \,.\, \nabla\mathbf{v}_1 = -\tfrac{1}{4}r(\phi^2 e^{2i\omega t} + \bar{\phi}^2 e^{-2i\omega t} + 2\phi\bar{\phi})\,\hat{\mathbf{r}}. \quad (4.11)$$

Hence on substituting from (4.8) and (4.11) into (2.21), we obtain for the steady field the equations

$$-\tfrac{1}{2}\rho r\phi\bar{\phi}\hat{\mathbf{r}} = -\nabla p_2^{(0)} + \mu_0\nabla^2\mathbf{v}_2^{(0)} + (\beta_1 + \tfrac{1}{2}\beta_2)\,r^2 d(\phi'\bar{\phi}')/dz\,\hat{\mathbf{z}} - \tfrac{1}{2}\beta_2 r\phi'\bar{\phi}'\hat{\mathbf{r}} \quad (4.12)$$

and

$$\nabla \,.\, \mathbf{v}_2^{(0)} = 0 \quad (4.13)$$

with the boundary conditions

$$\mathbf{v}_2^{(0)} = 0 \quad \text{on} \quad z = 0, d. \quad (4.14)$$

The system admits the representation of the flow field

$$\mathbf{v}_2^{(0)} = u(r,z)\,\hat{\mathbf{r}} + w(r,z)\,\hat{\mathbf{z}}, \quad p_2^{(0)} = p(r,z), \tag{4.15}$$

whereupon (4.12)–(4.14) become

$$\mu_0\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right] = \frac{\partial p}{\partial r}-\tfrac{1}{2}\rho r\phi\overline{\phi}+\tfrac{1}{2}\beta_2 r\phi'\overline{\phi}', \tag{4.16}$$

$$\mu_0\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right] = \frac{\partial p}{\partial z}-(\beta_1+\tfrac{1}{2}\beta_2)\,r^2\frac{d}{dz}(\phi'\overline{\phi}'), \tag{4.17}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0, \tag{4.18}$$

$$u = w = 0 \quad \text{on} \quad z = 0, d. \tag{4.19}$$

The system of equations (4.16)–(4.19) admits a solution of the form

$$u(r,z) = rf'(z), \quad w(r,z) = -2f(z), \quad p(r,z) = r^2 g(z)+h(z) \tag{4.20}$$

provided that the functions f, g and h satisfy the equations

$$\mu_0 f''' = 2g-\tfrac{1}{2}\rho\phi\overline{\phi}+\tfrac{1}{2}\beta_2\phi'\overline{\phi}', \tag{4.21}$$

$$-2\mu_0 f'' = h', \tag{4.22}$$

$$0 = g'-(\beta_1+\tfrac{1}{2}\beta_2)\,d(\phi'\overline{\phi}')/dz \tag{4.23}$$

and the boundary conditions

$$f = f' = 0 \quad \text{on} \quad z = 0, d. \tag{4.24}$$

The pressure functions $g(z)$ and $h(z)$ can be determined from (4.22) and (4.23) to within constants of integration k_1 and k_2:

$$g = \tfrac{1}{2}k_1+(\beta_1+\tfrac{1}{2}\beta_2)\,\phi'\overline{\phi}' \tag{4.25}$$

and

$$h = k_2-2\mu_0 f'. \tag{4.26}$$

Substituting from (4.25) into (4.21) we obtain

$$\mu_0 f''' = k_1+(2\beta_1+\tfrac{3}{2}\beta_2)\,\phi'\overline{\phi}'-\tfrac{1}{2}\rho\phi\overline{\phi}. \tag{4.27}$$

From (3.9) we find that

$$\phi\overline{\phi} = \frac{\cosh\Omega_r(1-\eta)-\cos\Omega_i(1-\eta)}{\cosh\Omega_r-\cos\Omega_i} \tag{4.28}$$

and

$$\phi'\overline{\phi}' = \frac{\rho\omega}{|\mu|}\frac{\cosh\Omega_r(1-\eta)-\cos\Omega_i(1-\eta)}{\cosh\Omega_r-\cos\Omega_i}, \tag{4.29}$$

where we have set

$$\Omega_r = \operatorname{Re}\Omega, \quad \Omega_i = \operatorname{Im}\Omega, \quad \eta = z/d. \tag{4.30}$$

Equations (4.27) and (4.24) now take the form

$$\left.\begin{aligned}(\mu_0/d)^3 f_{\eta\eta\eta} &= k_1+A\cosh\Omega_r(1-\eta)+B\cos\Omega_i(1-\eta),\\ f = f_\eta &= 0 \quad \text{on} \quad \eta = 0, 1,\end{aligned}\right\} \tag{4.31}$$

where
$$A \equiv \frac{\rho}{|\mu|}\frac{(2\beta_1+\frac{3}{2}\beta_2)\,\omega-\frac{1}{2}|\mu|}{\cosh\Omega_r-\cos\Omega_i}, \tag{4.32}$$

and
$$B \equiv \frac{\rho}{|\mu|}\frac{(2\beta_1+\frac{3}{2}\beta_2)\,\omega+\frac{1}{2}|\mu|}{\cosh\Omega_r-\cos\Omega_i}. \tag{4.33}$$

The solution of (4.31) is found to be

$$(\mu_0/d^3)f_\eta = \frac{A}{\Omega_r^2}\left\{-(1-3\eta)(1-\eta)\cosh\Omega_r-6\eta(1-\eta)\frac{\sinh\Omega_r}{\Omega_r}+\cosh\Omega_r(1-\eta)+\eta(2-3\eta)\right\}$$
$$+\frac{B}{\Omega_i^2}\left\{(1-3\eta)(1-\eta)\cos\Omega_i+6\eta(1-\eta)\frac{\sin\Omega_i}{\Omega_i}-\cos\Omega_i(1-\eta)-\eta(2-3\eta)\right\} \tag{4.34}$$

with
$$k_1 = \frac{6A}{\Omega_r^3}\{-\Omega_r\cosh\Omega_r+2\sinh\Omega_r-\Omega_r\}+\frac{6B}{\Omega_i^3}\{\Omega_i\cos\Omega_i-2\sin\Omega_i+\Omega_i\}. \tag{4.35}$$

Equation (4.34) gives the radial component of the time-independent flow velocity; the corresponding axial component can be obtained by integration of (4.34), but we shall not write it down here. For future reference, however, we note that differentiation of (4.34) gives

$$(\mu_0/d^3)f_{\eta\eta} = \frac{A}{\Omega_r^2}\left\{(4-6\eta)\cosh\Omega_r-(6-12\eta)\frac{\sinh\Omega_r}{\Omega_r}-\Omega_r\sinh\Omega_r(1-\eta)+2-6\eta\right\}$$
$$+\frac{B}{\Omega_i^2}\left\{-(4-6\eta)\cos\Omega_i+(6-12\eta)\frac{\sin\Omega_i}{\Omega_i}-\Omega_i\sin\Omega_i(1-\eta)-2+6\eta\right\}, \tag{4.36}$$

which is proportional to the steady component of shear stress in the fluid.

5. Flow reversal

In a Newtonian fluid the steady radial–axial flow is driven by centrifugal effects. This flow is therefore radially outward near the oscillating boundary and, by continuity, radially inward near the stationary boundary. The radial velocity changes sign exactly once in the interval $0 < \eta < 1$. Axial motion is towards the moving boundary and does not come to rest for any value of η in $0 < \eta < 1$, although this can happen when both planes oscillate (cf. Rosenblat 1960).

For the sake of completeness, and with a view to later comparison with non-Newtonian behaviour, we shall verify the above statements on the basis of the results obtained in the preceding section. When the fluid is Newtonian, we have

$$\mu = \mu_0, \quad \beta_1 = \beta_2 = 0, \tag{5.1}$$

$$\Omega = (4i\rho\omega d^2/\mu_0)^{\frac{1}{2}} = \Lambda(1+i), \quad \text{say}, \tag{5.2}$$

and
$$-A = B = \rho/2(\cosh\Lambda-\cos\Lambda) > 0 \quad \text{when} \quad \Lambda > 0. \tag{5.3}$$

Equation (4.36) can now be written in the form

$$H(\eta) \equiv (\mu_0\Lambda^2/Bd^3)f_{\eta\eta} = \Lambda[\sinh\Lambda(1-\eta)-\sin\Lambda(1-\eta)]-(4-6\eta)(\cosh\Lambda+\cos\Lambda)$$
$$+(6-12\eta)(\sinh\Lambda+\sin\Lambda)/\Lambda-4+12\eta. \tag{5.4}$$

From this we see that

$$H(0) = \Lambda(\sinh\Lambda - \sin\Lambda) - 4(\cosh\Lambda + \cos\Lambda) + 6\left(\frac{\sinh\Lambda + \sin\Lambda}{\Lambda}\right) - 4$$
$$= \sum_{n=1}^{\infty} \frac{(2n-2)(2n-1)[1+(-1)^n]}{(2n+1)!}\Lambda^{2n} > 0 \quad \text{when} \quad \Lambda > 0. \tag{5.5}$$

Hence $f_{\eta\eta}(0) > 0$, which means that in the neighbourhood of the oscillating plane $\eta = 0$ the radial flow is always outwards from the axis. Similarly

$$H(1) = 2(\cosh\Lambda + \cos\Lambda) - 6\left(\frac{\sinh\Lambda + \sin\Lambda}{\Lambda}\right) + 8$$
$$= 2\sum_{n=1}^{\infty} \frac{(2n-2)[1+(-1)^n]}{(2n+1)!}\Lambda^{2n} > 0 \quad \text{when} \quad \Lambda > 0. \tag{5.6}$$

This implies that $f_{\eta\eta}(1) > 0$, so that the flow is always radially inward near the stationary plane $\eta = 1$. Hence it follows that the radial velocity component changes sign at least once in the interior between the boundaries.

On the other hand we see from (5.4) that $H'(\eta) = 0$ if and only if

$$\Lambda^2[\cosh\Lambda(1-\eta) - \cos\Lambda(1-\eta)] = 6(\cosh\Lambda + \cos\Lambda) + 12(\sinh\Lambda - \sin\Lambda - \Lambda)/\Lambda. \tag{5.7}$$

It is easy to show that the left-hand side of (5.7) is a strictly monotonic function of η. Hence for each $\Lambda > 0$ there can be at most one value of η at which $H'(\eta) = 0$; that is, at most one stationary point of $H(\eta)$, and therefore at most one point where the radial velocity component changes sign. These arguments combine to verify that the radial velocity has just one change of sign in the interior between the planes.

We turn now to the non-Newtonian case, and begin by considering the behaviour of the flow at very small frequencies. In the limit $\omega \to 0$ we have that

$$\mu \to \mu_0, \quad \beta_1 \to \alpha_1, \quad \beta_2 \to \alpha_2 \tag{5.8}$$

and

$$\Omega \to \Lambda(1+i), \quad \text{where} \quad \Lambda = O(\omega). \tag{5.9}$$

It follows that, to leading order,

$$-A \approx B \approx \rho/\Lambda^2. \tag{5.10}$$

Substituting into (4.34) we find for the radial velocity component

$$(\mu_0/d^3) f_\eta = \tfrac{1}{60}\rho\eta(1-\eta)(6 - 15\eta + 5\eta^2), \tag{5.11}$$

in which the material constants α_1 and α_2 do not appear. Thus (5.11) has the same structure as the Newtonian flow field, and there is no flow reversal in the low-frequency limit.

It is not difficult to confirm that (5.11) also represents the secondary radial flow which occurs when the plane $\eta = 0$ rotates *steadily* about its axis with small angular velocity. In other words, the zero-frequency limit is the solution for steady rotation. Since (5.11) displays no reversal of the direction of the flow we have the important conclusion that flow reversal, if it takes place, is a consequence of the time dependence of the primary flow.

At first sight it might appear that this conclusion contradicts the calculations of Kramer & Johnson (1972), who find reversal of the radial flow when the primary azimuthal flow is time independent. However, Kramer & Johnson compute a solution for a fluid confined within a cylinder of finite radius, and it is the presence of the lateral boundary which explains the apparent discrepancy. In fact Kramer & Johnson (1972, p. 211) indicate that, if all other quantities are held fixed, flow reversal does not occur unless the cylinder radius is sufficiently small. This establishes a consistency between their results and ours, since the latter relate in effect to a cylinder of infinite radius.

We consider next the case of very large frequency. Standard methods can be used to compute asymptotic approximations to various quantities in the limit $\omega \to \infty$. Thus, integrating (3.8) by parts, we find

$$\mu = \zeta(0)/i\omega + O(\omega^{-2}). \tag{5.12}$$

Hence (3.10) becomes

$$\Omega = (4i\rho\omega d^2)^{\frac{1}{2}}\,[\zeta(0)/i\omega + O(\omega^{-2})]^{-\frac{1}{2}} = 2i\omega d(\rho/\zeta(0))^{\frac{1}{2}}\,[1 + O(\omega^{-1})], \tag{5.13}$$

which implies that $\quad \Omega_r = O(1), \quad \Omega_i = O(\omega) \quad \text{as} \quad \omega \to \infty. \quad (5.14)$

Integrating (4.9) by parts we obtain

$$\beta_1 = -\omega^{-2}\zeta(0) + O(\omega^{-3}), \tag{5.15}$$

and similarly (4.10) gives $\quad \beta_2 = \omega^{-2}\gamma(0,0) + O(\omega^{-3}). \quad (5.16)$

Substituting (5.12), (5.15) and (5.16) into (4.32) and (4.33) we find

$$A = \frac{\rho}{2\zeta(0)}\frac{3\gamma(0,0) - 5\zeta(0)}{\cosh\Omega_r - \cos\Omega_i} + O\left(\frac{1}{\omega}\right) \tag{5.17}$$

and

$$B = \frac{\rho}{2\zeta(0)}\frac{3\gamma(0,0) - 3\zeta(0)}{\cosh\Omega_r - \cos\Omega_i} + O\left(\frac{1}{\omega}\right). \tag{5.18}$$

These formulae, together with the estimates (5.14), show that the second term in (4.36) is small compared with the first term as $\omega \to \infty$. Thus (4.36) can be written in the form

$$(\mu_0/d^3)f_{\eta\eta} = -AU(\eta) + O(\omega^{-1}), \tag{5.19}$$

where A is now given by (5.17) and

$$U(\eta) \equiv \Omega_r^{-2}\{(6\eta - 4)\cosh\Omega_r + (6 - 12\eta)\sinh(\Omega_r)/\Omega_r + \Omega_r \sinh(1-\eta) - 2 + 6\eta\}. \tag{5.20}$$

Now

$$U(0) = \Omega_r^{-2}\{-4\cosh\Omega_r + 6\sinh(\Omega_r)/\Omega_r + \Omega_r\sinh\Omega_r - 2\}$$

$$= 2\sum_{n=1}^{\infty}\frac{(n-1)(2n-1)\Omega_r^{2n-2}}{(2n-1)!} > 0 \quad \text{when} \quad \Omega_r \neq 0; \tag{5.21}$$

similarly

$$U(1) = 2\Omega_r^{-2}\{\cosh\Omega_r - 3\sinh(\Omega_r)/\Omega_r + 2\}$$

$$= 4\sum_{n=1}^{\infty}\frac{(n-1)\Omega_r^{2n-2}}{(2n+1)!} > 0 \quad \text{when} \quad \Omega_r \neq 0. \tag{5.22}$$

Also, we have from (5.20) that

$$U'(\eta) = \Omega_r^{-2}\{6\cosh\Omega_r - 12\sinh(\Omega_r)/\Omega_r - \Omega_r^2\cosh\Omega_r(1-\eta) + 6\}, \tag{5.23}$$

and it is again the case that $U'(\eta) = 0$ at just one point in the interval $0 < \eta < 1$.

These results imply that the direction of the radial flow is determined completely by the sign of A. When $A < 0, f_{\eta\eta}(0) > 0, f_{\eta\eta}(1) > 0$ and the flow has the same direction as for a Newtonian fluid. When $A > 0, f_{\eta\eta}(0) < 0, f_{\eta\eta}(1) < 0$ and the whole flow field is in the opposite direction. We see now from (5.17) that flow reversal will occur at very high frequencies in and only in materials for which

$$\gamma(0,0) > \tfrac{5}{3}\zeta(0). \tag{5.24}$$

This result appears to be somewhat at variance with the conclusions of Bhatnagar & Rajeswari (1962) and Frater (1964). The calculations performed by the former authors suggest that flow reversal always occurs at very high frequencies, while Frater found that the radial flow is anti-Newtonian at intermediate frequencies but reverts to being Newtonian in the large-frequency limit. We presume that the discrepancies are due to the different constitutive equations used.

We proceed now to examine the characteristics of the flow for the whole range of frequencies. In the general case it is convenient to write (4.36) in the form

$$(\mu_0/d^3)f_{\eta\eta} = -AU(\eta) + BV(\eta), \tag{5.25}$$

where $U(\eta)$ is defined by (5.20), A and B by (4.32) and (4.33), and $V(\eta)$ by the formula

$$V(\eta) = \Omega_i^{-2}\{-(4-6\eta)\cos\Omega_i + (6-12\eta)\sin(\Omega_i)/\Omega_i - \Omega_i\ \sin\Omega_i(1-\eta) - 2 + 6\eta\}. \tag{5.26}$$

Using these formulae we define

$$Q_0 \equiv (2\mu_0/\rho d^3)f_{\eta\eta}(0) = \frac{M[V(0)-U(0)] + V(0) + U(0)}{\cosh\Omega_r - \cos\Omega_{ir}} \tag{5.27}$$

and

$$Q_1 \equiv (2\mu_0/\rho d^3)f_{\eta\eta}(1) = \frac{M[V(1)-U(1)] + V(1) + U(1)}{\cosh\Omega_r - \cos\Omega_i}, \tag{5.28}$$

where

$$M \equiv (4\beta_1 + 3\beta_2)\,\omega/|\mu|. \tag{5.29}$$

Evidently

$$Q_0 = 0, \quad Q_1 = 0 \tag{5.30}$$

are respectively the conditions that the flow changes direction in the neighbourhood of the plane $\eta = 0$ and the plane $\eta = 1$. The flow is Newtonian when $Q_0 > 0$ and $Q_1 > 0$ and anti-Newtonian when the inequalities are reversed.

For a given fluid (5.27)–(5.30) may be regarded as equations that determine the frequencies at which flow reversal occurs near the respective boundaries. It is of course not possible to solve these equations without providing details of the material functions μ, β_1 and β_2.

We have already seen that the flow is in the Newtonian direction at very low frequencies. As the frequency increases from zero various possibilities arise: the flow may not reverse direction for any value of the frequency, or it may reverse direction throughout the whole fluid at some frequency, or it may change direction near one of the boundaries only. In the last case, if reversal is at the stationary plane, there will be radial ouflow near both boundaries and a compensating inflow in the interior. If the reversal is at the oscillating boundary, there will be radial inflow near both boundaries and radial outflow in the interior. As the frequency is increased still further, reversal may take place at the other plane, and then the whole flow pattern will be anti-Newtonian. Moreover, since (5.30) are transcendental equations it is conceivable that many flow reversals can occur, in various regions of the fluid, as the frequency changes.

To gain some insight into possible behaviour we shall examine in detail the solution for specific forms of μ, β_1 and β_2. We choose the simplest Maxwell-type approximation for the material function $\zeta(s)$, namely

$$\zeta(s) = -(\mu_0^2/\alpha_1)\exp(\mu_0 s/\alpha_1) \quad (\alpha_1 < 0), \tag{5.31}$$

which is consistent with (2.17). It is convenient to introduce a Deborah number θ, the ratio of the relaxation time of the material to the period of the forced oscillation, defined by

$$\theta = -\omega\alpha_1/\mu_0. \tag{5.32}$$

Then substituting (5.31) and (5.32) into (3.8) and (4.9) respectively we obtain

$$\mu = \mu_0/(1+i\theta) \tag{5.33}$$

and

$$\beta_1 = \alpha_1/(1+\theta^2). \tag{5.34}$$

We note also that (3.10) now gives

$$\Omega_r = q\theta^{\frac{1}{2}}(1+\theta^2)^{\frac{1}{4}}\cos\chi, \quad \Omega_i = q\theta^{\frac{1}{2}}(1+\theta^2)^{\frac{1}{4}}\sin\chi, \tag{5.35}$$

where

$$q = (-4\rho d^2/\alpha_1)^{\frac{1}{2}} \tag{5.36}$$

and

$$\chi = \tfrac{1}{2}\tan^{-1}\theta + \tfrac{1}{4}\pi. \tag{5.37}$$

For the material function $\gamma(s_1, s_2)$ we assume a one-term approximate form analogous to (5.31), namely

$$\gamma(s_1, s_2) = a_2\vartheta^2\exp[-\vartheta(s_1+s_2)]. \tag{5.38}$$

This satisfies the condition (2.17) for any $\vartheta > 0$; hence ϑ is a free constant which is a material property. Substituting (5.38) into (4.10) we obtain

$$\beta_2 = \alpha_2\kappa^2/(\kappa^2+\theta^2), \tag{5.39}$$

where

$$\kappa^2 = (-\alpha_1\vartheta/\mu_0)^2. \tag{5.40}$$

The solution of (5.30) is in effect a four-parameter problem, the parameters being θ, q, κ^2 and λ, where

$$\lambda = -\alpha_2/\alpha_1. \tag{5.41}$$

Of these only θ incorporates the frequency and may therefore be regarded as a measure of the latter; q has the form of a Reynolds number, while κ^2 and λ are strictly material parameters. The functions U and V appearing in (5.27) and (5.28) depend only on θ and q, while M, defined by (5.29), can now be written as

$$M = \theta(1+\theta^2)^{\frac{1}{2}}\left[\frac{3\lambda\kappa^2}{\kappa^2+\theta^2} - \frac{4}{1+\theta^2}\right] \tag{5.42}$$

and is independent of q.

The low-frequency limit discussed earlier in this section is represented in the present model by the condition

$$\theta \ll 1; \tag{5.43}$$

this can be seen, for example, from (5.33), (5.43) and (5.39). We expect that no flow reversal will take place when (5.43) holds. On the other hand, the high-frequency limit contained in the inequality (5.24) now takes the form

$$\lambda\kappa^2 > \tfrac{5}{3}. \tag{5.44}$$

We expect that flow reversal can take place at high frequencies only if (5.44) holds.

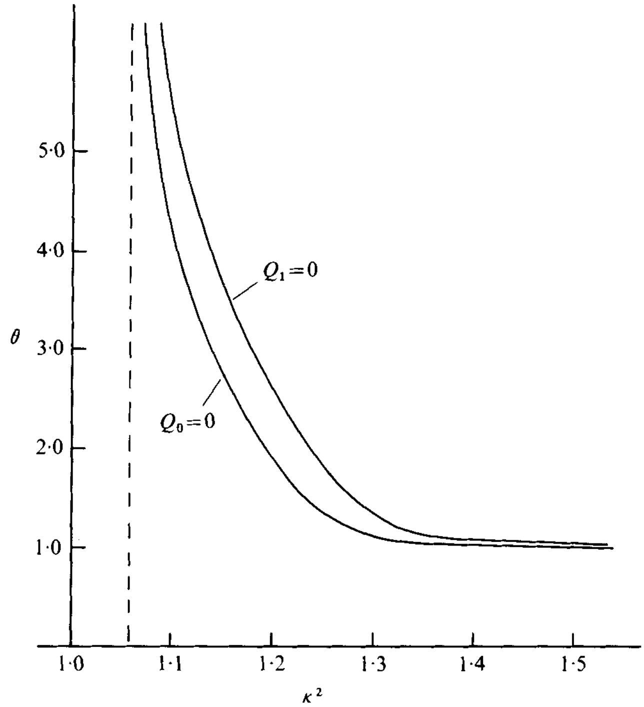

FIGURE 1. Points of flow reversal at moving and stationary boundaries. θ as a function of κ^2, with $q = 1{\cdot}0$ and $\lambda = 1{\cdot}5$.

Computations have been performed to discover if and when the quantities Q_0 and Q_1 change sign; after testing over a considerable range of values of all four parameters, we have been able to arrive at the following verifications of the low- and high-frequency conditions (5.43) and (5.44):

(i) Flow reversal does not occur for any q, λ or κ^2 if (5.43) holds.

(ii) Flow reversal does not occur for any q, λ, κ^2 or θ unless (5.44) holds. In particular this shows that the high-frequency condition (5.44) is actually a universal condition. Notice that if the stress–strain relation for the fluid is linear ($\alpha_2 = 0$) there can be no flow reversal

Our calculations for the model under consideration have also revealed the following features:

(iii) Given (5.43) and (5.44), reversal of the flow always take place once and only once at each boundary as θ changes with the other parameters held fixed.

(iv) Typically flow reversal takes place first (in the sense of θ increasing) at the moving boundary, and then at the stationary boundary. As mentioned earlier, this means that there is a range of frequencies for which there is radial inflow near both boundaries, and radial outflow in the interior. It appears, however, that the larger the value of $\lambda\kappa^2$, the closer to simultaneity is the reversal of the flow direction at the two boundaries.

These characteristics are illustrated in figure 1, which shows the values of θ at which flow reversal occurs as a function of κ^2, for fixed q and λ. For this particular computation we have taken $q = 1{\cdot}0$ and $\lambda = 1{\cdot}5$. The parameter q depends on the distance between the planes, and therefore can be adjusted more or less at will. The value $\lambda = 1{\cdot}5$ is not untypical for materials such as S.T.P.

If one takes the view that the material parameters μ_0, α_1 and α_2 are known from steady-flow experiments, the two values of θ at which the flow reverses at the boundaries can be regarded as determining the remaining parameter ϑ and testing the validity of the Maxwell-type model. Unfortunately there do not appear to be any observations of flow reversal in a time-periodic situation, other than those of Chang & Schowalter (1974), which are for a different configuration.

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